GENERALIZATION OF THE AIRY FUNCTION AND THE OPERATIONAL METHODS

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ABSTRACT. In this brief note the operatorial methods are applied to the study of the Airy function and its generalizations.

In this note we will discuss a method which can be usefully applied to the study of the Airy function. Before entering the details of the method we consider the following integral

$$C(\alpha,\beta) \,=\, \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{\imath\,\xi^\alpha}\,\xi^\beta\;, \tag{1}$$

which reduces to the ordinary Fresnel integral for $\alpha = 2$, $\beta = 0$. The use of standard analytical procedures allows to derive for it the following explicit expression in terms of the Gamma function

$$C(\alpha,\beta) \,=\, \frac{1}{\alpha}\,\Gamma\left(\frac{1+\beta}{\alpha}\right)\,\exp\left\{\imath\,\frac{\pi}{2}\,\frac{1+\beta}{\alpha}\right\}\;, \tag{2}$$

that will play a key role in the following.

Let us now consider the following integral transform

$$T(x|\alpha) \,=\, \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{\imath\,\xi^\alpha}\,f(x\,\xi) \tag{3}$$

which, on account of the operational identity [1]

$$\mathrm{e}^{\lambda\,x\,\partial_x}\,f(x) \,=\, f(\mathrm{e}^\lambda\,x)\;, \tag{4}$$

can be written as [2]

$$T(x|\alpha) \,=\, \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{\imath\,\xi^\alpha}\,\xi^{x\,\partial_x}\,f(x) \,=\, \hat{C}(\alpha, x\,\partial_x)\,f(x)\;, \tag{5}$$

where we have assumed that the integral in eq. (1) formally holds also when β is replaced by an operator (the integral itself is an operator). If the function $f(x)$ admits the expansion

$$f(x) \,=\, \sum_{n=0}^\infty a_n\,x^n\;, \tag{6}$$

we obtain (see ref. [2])

$$T(x|\alpha) \,=\, \sum_{n=0}^\infty a_n\,C(\alpha,n)\,x^n \tag{7}$$

which provides an appropriate series expansion for the integral transform in eq. (3).

The Airy function is defined through the expression [3]

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\xi \, \exp\left(\imath \frac{\xi^3}{3} + \imath \, x \, \xi \right) \tag{8}$$

which is easily shown to satisfy the differential equation

$$y'' - x \, y = 0 \, . \tag{9}$$

According to eqs. (5) and (7) we can expand the Airy function as follows

$$\begin{aligned} \mathrm{Ai}(x) &= \frac{\sqrt[3]{3}}{\pi} \Re \left\{ \int_0^{\infty} \mathrm{d}t \, \mathrm{e}^{\imath \, t^3} \left(\sqrt[3]{3} \, t \right)^{x \, \partial_x} \mathrm{e}^{\imath \, x} \right\} \\ &= \frac{1}{\sqrt[3]{9} \, \pi} \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{n+1}{3} \right) \cos\left(\frac{4n+1}{6} \, \pi \right) (\sqrt[3]{3} \, x)^n \, . \end{aligned} \tag{10}$$

(For further comments on earlier derivation see ref. [3]).

In the past, generalizations of the Airy function satisfying, for example, equations of the type

$$y'' + c_n \, x^n \, y = 0 \, . \tag{11}$$

have been proposed by Watson [4]. We consider first the example

$$\mathrm{Ai}_4(x) = \int_0^{\infty} \mathrm{d}t \, \cos\left(t^4 + 2 \, x \, t + 2 \, x^2 \right) \tag{12}$$

which, on account of the previously outlined procedure, can be cast in the form

$$\begin{aligned} \mathrm{Ai}_4(x) &= \Re \left\{ \mathrm{e}^{\imath \, 2 \, x^2} \, \hat{C}(4, x \, \partial_x) \, \mathrm{e}^{\imath \, 2 \, x} \right\} \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(\frac{n+1}{4} \right) \left\{ \cos(2 \, x^2) \, \cos\left(\frac{5n+1}{8} \, \pi \right) \right. \\ &\qquad \left. - \, \sin(2 \, x^2) \, \cos\left(\frac{5n+1}{8} \, \pi \right) \right\} (2 \, x)^n \, . \end{aligned} \tag{13}$$

Another example is represented by the function defined by the following integral representation

$$P(x,y) = \int_0^{\infty} \mathrm{d}u \, \mathrm{e}^{\imath (u^4 + x \, u^2 + y \, u)} \tag{14}$$

introduced by Pearcey (see [3] and references therein) in the context of electromagnetic field theory. In this case, we obtain

$$\begin{aligned} P(x,y) &= \hat{C}(4, 2 \, x \, \partial_x + y \, \partial_y) \, \mathrm{e}^{\imath \, (x + y)} \\ &= \frac{\mathrm{e}^{\imath \, \pi/8}}{4} \sum_{n=0}^{\infty} \mathrm{e}^{\imath \, 3 \, n \, \pi/4} \, x^n \sum_{k=0}^{n} \frac{1}{k! \, (n-k)!} \, \Gamma\left(\frac{2n-k+1}{4} \right) \left(\mathrm{e}^{-\imath \, \pi/8} \, \frac{y}{x} \right)^k \, . \end{aligned} \tag{15}$$

It is interesting to note that $P(x,y)$ satisfies a Schrödinger-like equation

$$\imath \, \partial_x \, P(x,y) = \partial_y^2 \, P(x,y) \, , \tag{16}$$

and, therefore, we can write

$$P(x,y) = \mathrm{e}^{-\imath \, x \, \partial_y^2} \, P(0,y) \, . \tag{17}$$

This result allows to write an alternative series expansion for $P(x, y)$. We first note that

$$P(0, y) = \hat{C}(4, y\,\partial_y)\,\mathrm{e}^{\imath\, y} = \frac{\mathrm{e}^{\imath\,\pi/8}}{4}\sum_{n=0}^{\infty}\frac{1}{n!}\,\Gamma\left(\frac{n+1}{4}\right)\left(\mathrm{e}^{\imath\,5\,\pi/8}\,y\right)^n . \tag{18}$$

Moreover, from eq. (17) and the operational identity defining the generalized Hermite polynomials [5]

$$\mathrm{e}^{w\,\partial_z^2}\,z^n = H_n(z, w)\,, \qquad H_n(z, w) = n!\sum_{k=0}^{[n/2]}\frac{1}{(n-2k)!\,k!}\,z^{n-2k}\,w^k\,, \tag{19}$$

we, finally, get

$$P(x, y) = \frac{\mathrm{e}^{\imath\,\pi/8}}{4}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{\imath\,5\,n\,\pi/8}}{n!}\,\Gamma\left(\frac{n+1}{4}\right)H_n(y, -\imath\,x). \tag{20}$$

This brief note has been aimed at providing the possibility of treating Airy type integral in a unified way. A forthcoming, more extended, note will treat further relevant consequences.

References

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